Conformal derivative and conformal transports over $(\overline{L}_n, g)$ spaces

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Abstract

Transports preserving the angle between two contravariant vector fields but changing their lengths proportional to their own lengths are introduced as "conformal" transports and investigated over $(\overline{L}_n, g)$-spaces. They are more general than the Fermi-Walker transports. In an analogous way as in the case of Fermi-Walker transports a conformal covariant differential operator and its conformal derivative are defined and considered over $(\overline{L}_n, g)$-spaces. Different special types of conformal transports are determined inducing also Fermi-Walker transports for orthogonal vector fields as special cases. Conditions under which the length of a non-null contravariant vector field could swing as a homogeneous harmonic oscillator are established. The results obtained regardless of any concrete field (gravitational) theory could have direct applications in such types of theories.

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1 Introduction

The construction of a frame of reference for an accelerated observer by means of vector fields preserving their lengths and the angles between them under a Fermi-Walker transport [1], [2], could also be related to the description of the motion of the axes of a gyroscope in a space with different (not only by sign) contravariant and covariant affine connections and metrics [$(\overline{L}_n, g)$-space]. On the other side, the problem arises how can we describe the motion of vector fields preserving the angles between them but, at the same time, changing the length of every one of them proportionally to its own length. In special cases

of $(\overline{L}_n, g)$-spaces there are different solutions of this problem which induces the definition of a conformal transport.

Remark. The most notions, abbreviations, and symbols in this paper are defined in the previous papers [1], [2]. The reader is kindly asked to refer to one of them.

Definition 1. The *conformal transport* is a special type of transport (along a contravariant vector field) under which the change of the length of a contravariant vector field is proportional to the length itself and the angle between two contravariant vector fields does not change.

1.1 Conformal transports in M_n-, V_n-, and U_n-spaces

(a) In flat and (pseudo) Riemannian spaces without or with torsion (M_n-, V_n-, and U_n- spaces respectively, $\dim M_n = n$), $[\nabla_u g = 0$ for $\forall u \in T(M)]$ the mentioned above problem could be easily solved by means of a conformal (angle preserving) mapping leading to the notion of *metric conformal to a given metric* [3].

If we construct the length of ξ by the use of the metric g in a (for instance) V_n-space as $l_\xi = \mid g(\xi, \xi) \mid^{\frac{1}{2}}$ and by the use of the conformal to g metric $\widetilde{g} = e^{2\varphi}.g$, $\varphi = \varphi(x^k) \in C^r(M)$, $r \geq 1$, as $\widetilde{l}_\xi = \mid \widetilde{g}(\xi, \xi) \mid^{\frac{1}{2}} = e^\varphi.l_\xi$, then the rate of change $u\widetilde{l_\xi}$ of the length of ξ along a contravariant vector field u leads to the relation $u\widetilde{l_\xi} = (u\varphi).e^\varphi.l_\xi + e^\varphi.ul_\xi = (u\varphi).\widetilde{l}_\xi + e^\varphi.ul_\xi$. If $ul_\xi = 0$, then $u\widetilde{l_\xi} = (u\varphi).\widetilde{l}_\xi$ and therefore, the change $u\widetilde{l_\xi}$ of the length $\widetilde{l}_\xi$ is proportional to $\widetilde{l}_\xi$. This is the case when ξ fulfils the equation $\nabla_u \xi = 0$ determining a parallel transport of ξ along u because of the relation in a M_n-, V_n-, or U_n-space

$$ul_\xi = \pm \frac{1}{l_\xi}.g(\nabla_u \xi, \xi) \ , \quad l_\xi \neq 0 \ . \tag{1}$$

The change of the cosine of the angle $[\cos(\xi, \eta) = (l_\xi.l_\eta)^{-1}.g(\xi, \eta)]$ between two contravariant vector fields ξ and η is described in these types of spaces by the relation

$$u[\cos(\xi, \eta)] = \frac{1}{l_\xi.l_\eta}.[g(\nabla_u \xi, \eta) + g(\xi, \nabla_u \eta)] -$$

$$-[\frac{1}{l_\xi}.(ul_\xi) + \frac{1}{l_\eta}.(ul_\eta)].\cos(\xi, \eta) \ , \qquad l_\xi \neq 0 \ , \quad l_\eta \neq 0 \ . \tag{2}$$

Under the conditions for parallel transport ($\nabla_u \xi = 0$, $\nabla_u \eta = 0$) of ξ and η, it follows that $u[\cos(\xi, \eta)] = 0$. Since the cosine between ξ and η defined by the use of the conformal to g metric $\widetilde{g}$ as

$$\cos(\xi, \eta) = \frac{1}{\widetilde{l}_\xi.\widetilde{l}_\eta}.\widetilde{g}(\xi, \eta) = \frac{e^{2\varphi}.g_{ij}.\xi^i.\eta^j}{e^{2\varphi}. \mid g_{kl}.\xi^k.\xi^l \mid^{\frac{1}{2}} . \mid g_{mn}.\eta^m.\eta^n \mid^{\frac{1}{2}}} =$$

$$= \frac{g_{ij}.\xi^i.\eta^j}{\mid g_{kl}.\xi^k.\xi^l \mid^{\frac{1}{2}} . \mid g_{mn}.\eta^m.\eta^n \mid^{\frac{1}{2}}} = \frac{1}{l_\xi.l_\eta}.g(\xi,\eta) \tag{3}$$

does not change by the replacement of the metric g with the metric $\widetilde{g}$, it follows that $u[\cos(\xi,\eta)] = 0$ for $u\widetilde{l_\xi} = (u\varphi).\widetilde{l}_\xi$, and $u\widetilde{l_\eta} = (u\varphi).\widetilde{l}_\eta$. This means that *parallel transports in a given M_n-, V_n-, or U_n-space induce conformal transports in the corresponding conformal space.*

If $ul_\xi = 0$, $ul_\eta = 0$ are not valid, then the proportionality of $u\widetilde{l_\xi}$ to $\widetilde{l}_\xi$ and of $u\widetilde{l_\eta}$ to $\widetilde{l}_\eta$ is violated even in these spaces. This fact induces the problem of finding out transports different from the parallel transport ($\nabla_u\xi = 0$, $\nabla_u\eta = 0$) under which the angle between two contravariant vector fields does not change and at the same time the rate of change of the lengths of these vector fields is proportional to the corresponding length.

(b) In Weyl's spaces ($\overline{W}_n$-spaces) [$\nabla_u g = \frac{1}{n}.Q_u.g$ for $\forall u \in T(M)$, $S(dx^i, \partial_j) = f^i{}_j(x^k)$], it follows from the general relations for ul_ξ and $u[\cos(\xi,\eta)]$ in $(\overline{L}_n, g)$-spaces

$$ul_\xi = \pm\frac{1}{2.l_\xi}.[(\nabla_u g)(\xi,\xi) + 2.g(\nabla_u\xi,\xi)] \ , \qquad l_\xi \neq 0 \ , \tag{4}$$

$$u[\cos(\xi,\eta)] = \frac{1}{l_\xi.l_\eta}.[(\nabla_u g)(\xi,\eta) + g(\nabla_u\xi,\eta) + g(\xi,\nabla_u\eta)] -$$

$$-[\frac{1}{l_\xi}.(ul_\xi) + \frac{1}{l_\eta}.(ul_\eta)].\cos(\xi,\eta) \ , \qquad l_\xi \neq 0 \ , \quad l_\eta \neq 0 \ , \tag{5}$$

that under the conditions for parallel transport ($\nabla_u\xi = 0$, $\nabla_u\eta = 0$) of ξ and η

$$ul_\xi = \pm\frac{1}{2.l_\xi}.\frac{1}{n}.Q_u.g(\xi,\xi) = \frac{1}{2.n}.Q_u.l_\xi \ , \qquad ul_\eta = \frac{1}{2.n}.Q_u.l_\eta \ , \tag{6}$$

$$u[\cos(\xi,\eta)] = \frac{1}{n}.Q_u.\cos(\xi,\eta) - \frac{1}{n}.Q_u.\cos(\xi,\eta) = 0 \ . \tag{7}$$

Therefore, the *parallel transports in $\overline{W}_n$-spaces are at the same time conformal transports.*

(c) The case of $(\overline{L}_n, g)$-spaces with equal to zero trace-free part of the covariant derivative of the metric g is analogous to this of Weyl's spaces $\overline{W}_n$.

Remark. In $(\overline{L}_n, g)$-spaces [4] a conformal change of a line element $ds^2 = g_{\overline{ij}}.dx^i.dx^j = f^k{}_i.f^l{}_j.g_{kl}.dx^i.dx^j$ could be trivially induced by choosing $f^k{}_i = e^\varphi.g^k_i$ with $\varphi = \varphi(x^k) \in C^r(M)$ and g^i_j as components of the Kronecker symbol. For this special case of the action of a contraction operator S (called in this

case conformal contraction operator S), the cosine of the angle between two contravariant non-null vector fields ξ and η defined as $\cos(\xi,\eta) = (l_\xi . l_\eta)^{-1}.g(\xi,\eta)$ with $l_\xi = \mid g(\xi,\xi) \mid^{\frac{1}{2}}$ and $l_\eta = \mid g(\eta,\eta) \mid^{\frac{1}{2}}$ does not change under the action of the contraction operator $S : S(e^k(x^l), e_i(x^l)) = f^k{}_i(x^l)$ [as it is the case for the contraction operator $S = C : C(e^k(x^l), e_i(x^l)) = g^k_i$]

$$\cos(\xi,\eta) = \frac{g_{\overline{ij}}.\xi^i.\eta^j}{\mid g_{\overline{kl}}.\xi^k.\xi^l \mid^{\frac{1}{2}} . \mid g_{\overline{mn}}.\eta^m.\eta^n \mid^{\frac{1}{2}}} =$$

$$= \frac{e^{2\varphi}.g_{ij}.\xi^i.\eta^j}{e^{2\varphi}. \mid g_{kl}.\xi^k.\xi^l \mid^{\frac{1}{2}} . \mid g_{mn}.\eta^m.\eta^n \mid^{\frac{1}{2}}} = \frac{g_{ij}.\xi^i.\eta^j}{\mid g_{kl}.\xi^k.\xi^l \mid^{\frac{1}{2}} . \mid g_{mn}.\eta^m.\eta^n \mid^{\frac{1}{2}}} . \tag{8}$$

1.2 Extended covariant differential operator over $(\overline{L}_n, g)$-spaces

If one considers the transformation properties of the contravariant and covariant affine connections over a $(\overline{L}_n, g)$-space, he can prove that the components of every affine connection are determined up to the set of the components of mixed tensor fields with contravariant rank 1 and covariant rank 2, i. e. if $\nabla_{\partial_k}\partial_j = \Gamma^i_{jk}.\partial_i$ and Γ^i_{jk} are components of a contravariant affine connection Γ, and $\nabla_{\partial_k} dx^i = P^i_{jk}.dx^j$, where P^i_{jk} are components of a covariant affine connection P in a given (here co-ordinate) basis, then $\overline{\Gamma}{}^i{}_{jk}$ and $\overline{P}{}^i{}_{jk}$

$$\overline{\Gamma}{}^i{}_{jk} = \Gamma^i_{jk} - \overline{A}{}^i{}_{jk} , \qquad \overline{P}{}^i{}_{jk} = P^i_{jk} - \overline{B}{}^i{}_{jk} ,$$
$$\overline{A} = \overline{A}{}^i{}_{jk}.\partial_i \otimes dx^j \otimes dx^k , \quad \overline{B} = \overline{B}{}^i{}_{jk}.\partial_i \otimes dx^j \otimes dx^k , \quad \overline{A} , \overline{B} \in \otimes^1{}_2(M) ,$$

are components (in the same basis) of a new contravariant affine connection $\overline{\Gamma}$ and a new covariant affine connection $\overline{P}$ respectively.

$\overline{\Gamma}$ and $\overline{P}$ correspond to a new ["extended" with respect to ∇_u, $u \in T(M)$] covariant differential operator ${}^e\nabla_u$

$${}^e\nabla_{\partial_k}\partial_j = \overline{\Gamma}{}^i{}_{jk}.\partial_i , \qquad {}^e\nabla_{\partial_k} dx^i = \overline{P}{}^i{}_{jk}.dx^j ,$$

with the same properties as the covariant differential operator ∇_u.

Remark. On the grounds of the last relations the transformation properties of $\overline{\Gamma}{}^i{}_{jk}$ and $\overline{P}{}^i{}_{jk}$ can be proved in analogy to the proofs of the transformation properties of Γ^i_{jk} and P^i_{jk}.

If we choose the tensors $\overline{A}$ and $\overline{B}$ with certain predefined properties, then we can find $\overline{\Gamma}$ and $\overline{P}$ with predetermined characteristics. On the other side, $\overline{A}{}^i{}_{jk}$ and $\overline{B}{}^i{}_{jk}$ are related to each other on the basis of the commutation relations of ${}^e\nabla_u$ and ∇_u with the contraction operator S. From

$$\nabla_u \circ S = S \circ \nabla_u , \qquad {}^e\nabla_u \circ S = S \circ {}^e\nabla_u ,$$
$$S(\partial_j \otimes dx^i) = S(dx^i \otimes \partial_j) = S(dx^i, \partial_j) = f^i{}_j , \quad f_m{}^j.f^l{}_j = g^i_j , \quad \det(f^i{}_j) \neq 0 ,$$

after some computations [2] we have

$$\overline{A}^i{}_{jk} = - \overline{B}^m{}_{lk}.f^l{}_j.f_m{}^i = - \overline{B}^{\underline{i}}{}_{\overline{j}k} , \qquad \overline{B}^i{}_{jk} = - \overline{A}^l{}_{mk}.f^i{}_l.f_j{}^m = - \overline{A}^{\overline{i}}{}_{\underline{j}k} .$$

We can write ${}^e\nabla_{\partial_k}$ in the form

$${}^e\nabla_{\partial_k} = \nabla_{\partial_k} - \overline{A}_{\partial_k} \quad \text{with} \quad \overline{A}_{\partial_k} = \overline{A}^i{}_{jk}.\partial_i \otimes dx^j .$$

${}^e\nabla_u$ can also be written in the form

$${}^e\nabla_u = \nabla_u - \overline{A}_u \quad \text{with} \quad \overline{A}_u = \overline{A}^i{}_{jk}.u^k.\partial_i \otimes dx^j .$$

$\overline{A}_u$ appears as a mixed tensor field of second rank but acting on tensor fields as a covariant differential operator because ${}^e\nabla_u$ is defined as covariant differential operator with the same properties as the covariant differential operator ∇_u. In fact, $\overline{A}_u$ can be defined as $\overline{A}_u = \nabla_u - {}^e\nabla_u$. If $\overline{A}_u$ is a given mixed tensor field, then ${}^e\nabla_u$ can be constructed in a predetermined way.

In previous papers [1], [2], we have considered Fermi-Walker transports over (L_n, g) and $(\overline{L}_n, g)$-spaces leading to preservation of the lengths of two contravariant vector fields and the angle between them, when they are transported along a non-null (non-isotropic) contravariant vector field. The investigations have been based on a special form of the extended covariant differential operator ${}^e\nabla_u$ determined over a (L_n, g)- or a $(\overline{L}_n, g)$-space as ${}^e\nabla_u = \nabla_u - \overline{A}_u$, where ∇_u is the covariant differential operator:

$$\nabla_u : v \to \nabla_u v = \overline{v} , \qquad v, \ \overline{v} \in \otimes^k{}_l(M) .$$

u is a contravariant vector field, $u \in T(M)$, v and $\overline{v}$ are tensor fields with given contravariant rank k and covariant rank l.

In accordance to its property [1] $\overline{A}_{u+v} = \overline{A}_u + \overline{A}_v$, $\overline{A}_u$ has to be linear to u. The existence of covariant and contravariant metrics g and $\overline{g}$ in a $(\overline{L}_n, g)$-space allows us to represent $\overline{A}_u$ in the form $\overline{A}_u = \overline{g}(A_u)$. There are at least three possibilities for construction of a covariant tensor field of second rank A_u in such a way that A_u is linear to u, i. e. we can determine A_u as

A. $A_u = C(u) = A(u) = A_{ij\overline{k}}.u^k.dx^i \otimes dx^j$.

B. $A_u = C(u) = \nabla_u B = B_{ij;k}.u^k.dx^i \otimes dx^j$.

C. $A_u = C(u) = A(u) + \nabla_u B = (A_{ij\overline{k}} + B_{ij;k}).u^k.dx^i \otimes dx^j$.

These three possibilities for definition of A_u lead to three types (A, B, and C respectively) of the extended covariant differential operator ${}^e\nabla_u = \nabla_u - \overline{g}(A_u)$. It can obey additional conditions determining the structure of the mixed tensor field $\overline{A}_u = \overline{g}(A_u)$. One can impose given conditions on ${}^e\nabla_u$ leading to predetermined properties of $\overline{A}_u$ and vice versa: one can impose conditions on the tensor field $\overline{A}_u$ leading to predetermined properties of ${}^e\nabla_u$. The method of finding out the conditions for Fermi-Walker transports over $(\overline{L}_n, g)$-spaces allow us to consider other types of transports with important properties for describing the motion of physical systems over such type of spaces.

As it was mentioned above an interesting problem is the finding out of transports along a contravariant vector field under which the change of the length of a contravariant vector field is proportional to the length itself and, at the same time, the angle between two contravariant vector fields does not change. These preconditions for a transport lead to its definition and its applications as a conformal transport over $(\overline{L}_n, g)$-spaces.

In Sec. 2 conformal transports over $(\overline{L}_n, g)$-spaces are determined and considered with respect to their structure. A conformal covariant differential operator and its corresponding conformal derivative are introduced. In Sec. 3 Fermi-Walker transports along a contravariant vector field for orthogonal to it contravariant vector fields are found on the basis of the structure of conformal transports. In Sec. 4 conformal transports of null vector fields are discussed. In Sec. 5 the length of a contravariant vector field as homogeneous harmonic oscillator with given frequency is considered on the grounds of a conformal transport. Sec. 6 comprises some concluding remarks.

Remark. All formulas written in index-free form are identical and valid in their form (but not in their contents) for (L_n, g)- and $(\overline{L}_n, g)$-spaces. The difference between them appears only if they are written in a given (co-ordinate or non-co-ordinate) basis.

2 Conformal transports over $(\overline{L}_n, g)$-spaces

Let us now take a closer look at the notion of conformal transport over $(\overline{L}_n, g)$-spaces.

Remark. The following below conditions are introduced on the analogy of the case of Fermi-Walker transports.

The main assumption related to the notion of conformal transport and leading to a definition of an external covariant differential operator ${}^{e}\nabla_u = {}^{c}\nabla_u$ (called conformal covariant differential operator) is that the parallel transports ${}^{c}\nabla_u \xi = 0$ and ${}^{c}\nabla_u \eta = 0$ of two contravariant non-null vector fields ξ and η respectively induce proportional to the lengths $l_\xi = \mid g(\xi, \xi) \mid^{\frac{1}{2}}$ and $l_\eta = \mid g(\eta, \eta) \mid^{\frac{1}{2}}$ changes of l_ξ and l_η along a contravariant vector field u as well as preservation of the angle between ξ and η with respect to a special transport of the covariant differential operator ∇_u. The rate of change of the length l_ξ of a non–null contravariant vector field ξ and the rate of change of the cosine of the angle between two non-null contravariant vector fields ξ and η over a $(\overline{L}_n, g)$-space can be found in the forms (4) and (5). If ${}^{c}\nabla_u \xi = \nabla_u \xi - \overline{A}_u \xi = 0$ and ${}^{c}\nabla_u \eta = \nabla_u \eta - \overline{A}_u \eta = 0$, then the conditions have to be fulfilled

$$ul_\xi = \pm \frac{1}{2.l_\xi}.[(\nabla_u g)(\xi, \xi) + 2.g(\overline{A}_u \xi, \xi)] = f(u).l_\xi \ , \quad l_\xi \neq 0 \ , \quad f(u) \in C^r(M) \ , \tag{9}$$

$$u[\cos(\xi, \eta)] = \frac{1}{l_\xi.l_\eta}.[(\nabla_u g)(\xi, \eta) + g(\overline{A}_u \xi, \eta) + g(\xi, \overline{A}_u \eta)] -$$

$$-[\frac{1}{l_\xi}.(ul_\xi) + \frac{1}{l_\eta}.(ul_\eta)].\cos(\xi,\eta) = 0 \ . \tag{10}$$

From the first condition, under the assumption $\overline{A}_u = \overline{g}(A_u)$ with $A_u = C(u) = C_{ij}(u).dx^i \otimes dx^j$, we obtain

$$ul_\xi = \pm\frac{1}{2.l_\xi}.[(\nabla_u g)(\xi,\xi) + 2.g(\overline{g}(C(u))(\xi),\xi)] = f(u).l_\xi \ , \tag{11}$$

where [1]

$$\begin{array}{c} g(\overline{g}(C(u))(\xi),\xi) = [{}_sC(u)](\xi,\xi) = {}_sC_{jk}(u).\xi^{\overline{j}}.\xi^{\overline{k}} \ , \\ {}_sC(u) = C_{(jk)}(u).dx^j.dx^k \ , \quad C_{(jk)}(u) = \frac{1}{2}.[C_{jk}(u) + C_{kj}(u)] \\ dx^j.dx^k = \frac{1}{2}(dx^j \otimes dx^k + dx^k \otimes dx^j) \ . \end{array} \tag{12}$$

On the other hand, $(\nabla_u g)(\xi,\xi) = g_{jk;m}.u^m.\xi^{\overline{j}}.\xi^{\overline{k}}$. Therefore, the condition $ul_\xi = f(u).l_\xi$ for $\forall\xi \in T(M)$ leads to the relations

$$ul_\xi = \pm\frac{1}{2.l_\xi}.\{(\nabla_u g)(\xi,\xi) + 2.[{}_sC(u)](\xi,\xi)\} = f(u).l_\xi \ , \quad l_\xi \neq 0 \ , \quad f(u) \in C^r(M) \ ,$$

$$\begin{array}{c} (\nabla_u g)(\xi,\xi) + 2.[{}_sC(u)](\xi,\xi) = \pm 2.f(u).l_\xi^2 = 2.f(u).g(\xi,\xi) \text{ for } \forall\xi \in T(M) \ , \\ [\nabla_u g + 2.{}_sC(u) - 2.f(u).g](\xi,\xi) = 0 \text{ for } \forall\xi \in T(M) \ , \\ \nabla_u g = -2.{}_sC(u) + 2.f(u).g : \ {}_sC(u) = -\frac{1}{2}.\nabla_u g + f(u).g \ . \end{array} \tag{13}$$

Since $C(u)$ and respectively ${}_sC(u)$ have to be linear to u, $f(u)$ should have the form $f(u) = f_{\overline{k}}.u^k$, $f \in T^*(M)$. Therefore,

$${}_sC(u) = -\frac{1}{2}.\nabla_u g + f(u).g \ , \quad u \in T(M) \ , \quad f \in T^*(M) \ , \text{ for } \forall\xi \in T(M) \ . \tag{14}$$

By the use of the condition $ul_\xi = f(u).l_\xi$ for $\forall\xi \in T(M)$ and $l_\xi \neq 0$, we have found the explicit form of the symmetric part ${}_sC(u)$ of $C(u) = {}_sC(u) + {}_aC(u)$, where

$$\begin{array}{c} {}_aC(u) = C_{[jk]}(u).dx^j \wedge dx^k \ , \ C_{[jk]}(u) = \frac{1}{2}.[C_{jk}(u) - C_{kj}(u)] \ , \\ dx^j \wedge dx^k = \frac{1}{2}(dx^j \otimes dx^k - dx^k \otimes dx^j) \ . \end{array} \tag{15}$$

If we now assume the validity of the first condition $ul_\xi = f(u).l_\xi$ [fulfilled for ${}_sC(u) = -\frac{1}{2}.\nabla_u g + f(u).g$], then from the expression for $u[\cos(\xi,\eta)]$ we obtain the relations:

$$u[\cos(\xi,\eta)] = \frac{1}{l_\xi.l_\eta}.[(\nabla_u g)(\xi,\eta) + g(\overline{A}_u\xi,\eta) + g(\xi,\overline{A}_u\eta)] -$$

$$-[\frac{1}{l_\xi}.f(u).l_\xi+\frac{1}{l_\eta}.f(u).l_\eta].\cos(\xi,\eta)\ ,\quad \text{for } \forall\xi,\eta\in T(M)\ , \tag{16}$$

$$\begin{gathered}
g(\overline{A}_u\xi,\eta)=g(\overline{g}(C(u))(\xi),\eta)=g_{\overline{i}\overline{j}}.g^{il}.C_{\overline{l}\overline{k}}(u).\xi^k.\eta^j=g^l_j.C_{\overline{l}\overline{k}}(u).\xi^k.\eta^j=\\
=C_{\overline{j}\overline{k}}(u).\eta^j.\xi^k=C_{jk}(u).\eta^{\overline{j}}.\xi^{\overline{k}}=[C(u)](\eta,\xi)\ ,\\
g(\overline{A}_u\eta,\xi)=g(\overline{g}(C(u))(\eta),\xi)=g_{\overline{i}\overline{j}}.g^{il}.C_{\overline{l}\overline{k}}(u).\eta^k.\xi^j=g^l_j.C_{\overline{l}\overline{k}}(u).\eta^k.\xi^j=\\
=C_{\overline{j}\overline{k}}(u).\xi^j.\eta^k=C_{jk}(u).\xi^{\overline{j}}.\eta^{\overline{k}}=[C(u)](\xi,\eta)\ ,\\
(\nabla_u g)(\xi,\eta)+g(\overline{A}_u\xi,\eta)+g(\xi,\overline{A}_u\eta)=(\nabla_u g)(\xi,\eta)+[{}_sC(u)](\eta,\xi)+[{}_aC(u)](\eta,\xi)+\\
+[{}_sC(u)](\xi,\eta)+[{}_aC(u)](\xi,\eta)\ .
\end{gathered} \tag{17}$$

Since $[{}_sC(u)](\eta,\xi)=[{}_sC(u)](\xi,\eta)=-\frac{1}{2}.(\nabla_u g)(\xi,\eta)+f(u).g(\xi,\eta)$ and $[{}_aC(u)](\eta,\xi)=-[{}_aC(u)](\xi,\eta)$, it follows for $u[\cos(\xi,\eta)]$

$$u[\cos(\xi,\eta)]=\frac{1}{l_\xi.l_\eta}.\{(\nabla_u g)(\xi,\eta)+2.[{}_sC(u)](\xi,\eta)\}-2.f(u).\cos(\xi,\eta)=$$

$$=\frac{1}{l_\xi.l_\eta}.\{(\nabla_u g)(\xi,\eta)+2.f(u).g(\xi,\eta)-(\nabla_u g)(\xi,\eta)\}-2.f(u).\cos(\xi,\eta)=$$

$$=\frac{1}{l_\xi.l_\eta}.2.f(u).l_\xi.l_\eta.\cos(\xi,\eta)-2.f(u).\cos(\xi,\eta)=0\ ,\quad g(\xi,\eta)=l_\xi.l_\eta.\cos(\xi,\eta)\ . \tag{18}$$

Therefore, $C(u)$ will have the explicit form

$$C(u)={}_aC(u)-\frac{1}{2}.\nabla_u g+f(u).g\ . \tag{19}$$

Now, we can define the notion of conformal covariant differential operator.

Definition 2. A *conformal covariant differential operator* ${}^c\nabla_u$ in a $(\overline{L}_n,g)$-space. An extended covariant operator ${}^e\nabla_u$ with the structure

$${}^e\nabla_u={}^c\nabla_u=\nabla_u-\overline{g}(C(u))\ ,$$

where

$$C(u)={}_aC(u)-\frac{1}{2}.\nabla_u g+f(u).g\ ,$$

$$C(u)\in\otimes_2(M),\qquad {}_aC(u)\in\Lambda^2(M)\ ,\quad u\in T(M)\ ,\quad f\in T^*(M)\ ,\quad g\otimes_{s2}(M)\ .$$

is called conformal covariant differential operator. It is denoted as ${}^c\nabla_u$.

Remark. For $f(u)=0:\ {}^c\nabla_u={}^F\nabla_u$, i. e. for $f(u)=0$ the conformal covariant differential operator ${}^c\nabla_u$ is identical with the Fermi covariant differential operator ${}^F\nabla_u$.

On the analogy of the case of ${}^F\nabla_u$ we can have three types (A, B, and C) of ${}^c\nabla_u$:

Table 1. Types of conformal covariant differential operator ${}^c\nabla_u$

Type of ${}^c\nabla_u$	Form of $C(u)$
A	$C(u) = A(u) = A_{ij\overline{k}}.u^k.dx^i \otimes dx^j$
B	$C(u) = \nabla_u B = B_{ij;k}.u^k.dx^i \otimes dx^j$
C	$C(u) = A(u) + \nabla_u B = (A_{ij\overline{k}} + B_{ij;k}).u^k.dx^i \otimes dx^j$

The first two types A and B are special cases of type C.

From the explicit form of $C(u) = {}_aC(u) - \frac{1}{2}\nabla_u g + f(u).g$ we can choose ${}_aC(u) = {}_aA(u)$ and ${}_sC(u) = {}_sA(u) + \nabla_u B$ with ${}_sA(u) = f(u).g$ and $B = -\frac{1}{2}.\nabla_u g$.

A type of general ansatz (without too big loss of generality) for $A(u) = {}_aA(u) + {}_sA(u) = {}_aA(u) + f(u).g$ (keeping in mind its linearity to u and the form of $\overline{A}_u = u^k.\overline{A}_{\partial_k}$) has the form [1]

$$A_{ij}(u) = p_{\overline{k}}.u^k.{}^F\omega_{ij} + f_{\overline{k}}.u^k.g_{ij} \ , \tag{20}$$

or

$$A(u) = p(u).{}^F\omega + f(u).g \ , \tag{21}$$

where

$$ {}^F\omega_{ij} = -{}^F\omega_{ji} \ , \quad {}_aA(u) = \frac{1}{2}.[A_{ij}(u) - A_{ji}(u)].dx^i \wedge dx^j \ , $$

$p, f \in T^*(M)$ are arbitrary given covariant vector fields, $p(u) = p_{\overline{k}}.u^k = S(p,u)$, $f(u) = f_{\overline{k}}.u^k = S(f,u)$, ${}^F\omega = {}^F\omega_{ij}.dx^i \wedge dx^j$ is an arbitrary given covariant antisymmetric tensor field of second rank. Therefore, $\overline{g}(A(u)) = \overline{g}(p(u).{}^F\omega) + \overline{g}(f(u).g) = p(u).\overline{g}({}^F\omega) + f(u).\overline{g}(g)$, $\overline{g}(g) = g^{\overline{ij}}.g_{jk}.\partial_i \otimes dx^k$. If we express p and f by the use of their corresponding with respect to the metric g contravariant vector fields $b = \overline{g}(p) : g(b) = p$, and $q = \overline{g}(f) : g(q) = f$ respectively, then $\overline{A}_u$ will obtain the form

$$\overline{A}_u = \overline{g}(C(u)) = g(b,u).\overline{g}({}^F\omega) + g(q,u).\overline{g}(g) - \frac{1}{2}.\overline{g}(\nabla_u g) \ , \quad b, q \in T(M) \ . \tag{22}$$

The components $\overline{A}^i{}_{jk}$ of $\overline{A}_{\partial_k}$ in a co-ordinate basis have the form

$$\overline{A}^i{}_{jk} = g^{im}.{}^F\omega_{\overline{m}j}.g_{\overline{kl}}.b^l + g_{\overline{kl}}.q^l.g^{im}.g_{\overline{m}j} - \frac{1}{2}.g^{i\overline{m}}.g_{mj;k} \ . \tag{23}$$

Respectively, the components $\overline{B}^i{}_{jk}$ of $\overline{B}_{\partial_k}$ in a co-ordinate basis have the form

$$\overline{B}^i{}_{jk} = -\overline{A}^{\overline{i}}{}_{\underline{j}k} = g^{\overline{i}m}.{}^F\omega_{\overline{m}\underline{j}}.g_{\overline{kl}}.b^l + g_{\overline{kl}}.q^l.g^{\overline{i}m}.g_{\overline{m}\underline{j}} - \frac{1}{2}.g^{\overline{im}}.g_{m\overline{j};k} \ . \tag{24}$$

We have now the free choice of the contravariant vector fields b and q, which could depend on the physical problem to be considered. For the determination

of Fermi-Walker transports the vector field b has been chosen as $b = \frac{1}{e}.u$ with $e = g(u,u) \neq 0$. There are other possibilities for the choice of b and q.

The conformal covariant differential operator will now have the form

$$^c\nabla_u = \nabla_u - \overline{A}_u = \nabla_u - [g(b,u).\overline{g}(^F\omega) + g(q,u).\overline{g}(g) - \frac{1}{2}.\overline{g}(\nabla_u g)] \ . \tag{25}$$

The result $^c\nabla_u v$ of the action of a conformal covariant differential operator $^c\nabla_u$ (of type A, B, and C) on a tensor field $v \in \otimes^k{}_l(M)$ is called *conformal derivative* of type A, B, and C respectively of the tensor field v.

3 Fermi-Walker transports for orthogonal to u vector fields

The free choice of the vector fields b and q allows us to determine another type of Fermi-Walker transport for orthogonal to u vector fields than the defined in [1].

(a) If we chose $b = \frac{1}{e}.u$ and $q = \xi$ in the expression for $^c\nabla_u\xi$ we will have

$$^c\nabla_u\xi = \nabla_u\xi - [\overline{g}(^F\omega)(\xi) + l.\xi - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi)] \ , \tag{26}$$

where $l = g(\xi,u)$, $\overline{g}(g)(\xi) = \overline{g}[g(\xi)] = \xi$. Then $^c\nabla_u\xi = 0$ is equivalent to a conformal transport in the form

$$\nabla_u\xi = [\overline{g}(^F\omega)(\xi) + l.\xi - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi)] \ . \tag{27}$$

It is obvious that if the vector field ξ is orthogonal to u $[l = g(u,\xi) = 0]$, then $\nabla_u\xi = [\overline{g}(^F\omega)(\xi) - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi)] = \overline{g}[^F\omega(\xi)] - \frac{1}{2}.\overline{g}[(\nabla_u g)(\xi)]$ is a generalized Fermi-Walker transport of type C along a non-null vector field u.

Remark. If $q = \xi$ (or $b = \xi$, or $b = q = \xi$) in the expression for $^c\nabla_u\xi$, then $^c\nabla_u\xi$ is not more a linear transport with respect to the contravariant vector field ξ.

(b) If we chose $b = q = \xi$ in the expression for $^c\nabla_u\xi$, it follows that

$$^c\nabla_u\xi = \nabla_u\xi - \{l.[\overline{g}(^F\omega)(\xi) + \xi] - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi)\} \ . \tag{28}$$

Then $^c\nabla_u\xi = 0$ is equivalent to a conformal transport in the form

$$\nabla_u\xi = l.[\overline{g}(^F\omega)(\xi) + \xi] - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi) \ . \tag{29}$$

For an orthogonal to u vector field ξ $[l = g(\xi,u) = 0]$ we obtain a Fermi-Walker transport of type B for the vector field ξ

$$\nabla_u\xi = -\frac{1}{2}.\overline{g}(\nabla_u g)(\xi) = \overline{g}[(\nabla_u B)(\xi)] \tag{30}$$

with $B = -\frac{1}{2}.g$.

In this case the condition $e = g(u,u) \neq 0$ is not used and u could be a non-null contravariant vector field ($e \neq 0$) as well as a null contravariant vector field ($e = 0$).

(c) One of the vector fields b and q or both vectors could also be related to the Weyl's vector in a $(\overline{L}_n, g)$-space. If we represent $\nabla_u g$ by means of its trace-free part and its trace part in the form

$$\nabla_u g = {}^s\nabla_u g + \frac{1}{n}.Q_u.g \ , \qquad \dim M = n \ , \tag{31}$$

where $\overline{g}[{}^s\nabla_u g] = 0$, $Q_u = \overline{g}[\nabla_u g] = g^{\overline{kl}}.g_{kl;j}.u^j = Q_j.u^j$, $Q_j = g^{\overline{kl}}.g_{kl;j}$.

Remark. The covariant vector $\overline{Q} = \frac{1}{n}.Q = \frac{1}{n}.Q_j.dx^j$ is called Weyl's vector. The operator ${}^s\nabla_u = \nabla_u - \frac{1}{n}.Q_u$ is called trace-free covariant operator.

(c_1) If we chose $b = \frac{1}{e}.u$ and $q = \overline{g}(\overline{Q}) = \widetilde{Q}$, then

$${}^c\nabla_u = \nabla_u - [\overline{g}({}^F\omega) + g(\widetilde{Q}, u).\overline{g}(g) - \frac{1}{2}.\overline{g}(\nabla_u g)] \ , \tag{32}$$

and ${}^c\nabla_u \xi = 0$ will have the form

$$\nabla_u \xi = \overline{g}[({}^F\omega)(\xi)] + g(\widetilde{Q}, u).\xi - \frac{1}{2}.\overline{g}[(\nabla_u g)(\xi)] \ . \tag{33}$$

If the vector field u is orthogonal to $\widetilde{Q}$, i. e. $g(\widetilde{Q}, u) = 0$, then $\nabla_u \xi = \overline{g}[({}^F\omega)(\xi)] - \frac{1}{2}.\overline{g}[(\nabla_u g)(\xi)]$ and we have a generalized Fermi-Walker transport of type C along a non-null vector field u.

(c_2) If we chose $b = q = \overline{Q}$ in the expression for ${}^c\nabla_u$, we will have the relation

$${}^c\nabla_u \xi = \nabla_u \xi - \{g(\widetilde{Q}, u).[\overline{g}({}^F\omega)(\xi) + \xi] - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi)\} \ . \tag{34}$$

Then ${}^c\nabla_u \xi = 0$ is equivalent to

$$\nabla_u \xi = g(\widetilde{Q}, u).[\overline{g}({}^F\omega)(\xi) + \xi] - \frac{1}{2}.\overline{g}(\nabla_u g)(\xi) \ . \tag{35}$$

For an orthogonal to $\widetilde{Q}$ vector field u $[g(\widetilde{Q}, u) = 0]$ we obtain a Fermi-Walker transport of type B for the vector field $\xi : \nabla_u \xi = -\frac{1}{2}.\overline{g}[(\nabla_u g)(\xi)]$ as in the case (b) for $l = 0$. Here u could also be a non-null ($e \neq 0$) or null ($e = 0$) contravariant vector field.

Remark. The assumption $g(\widetilde{Q}, u) = 0$ contradicts to the physical interpretation of $\widetilde{Q}$ as a vector potential A of the electromagnetic field in W_n-spaces ($n = 4$) [5] because of the relation $g(A, u) = e_0.[g(u,u)]/[g(R,u) \neq 0$. In our case here $\widetilde{Q}$ could be more related to the Lorentz force ${}_LF = \overline{g}[F(u)]$ than with A. F is the electromagnetic tensor: $F = dA = (A_{j,i} - A_{i,j}).dx^i \wedge dx^j$.

4 Conformal transports for null vector fields

$l_\xi^2 = 0$ is fulfilled for a null contravariant vector field ξ. Then the condition for a conformal transport of ξ

$$l_\xi . u l_\xi = \pm \frac{1}{2} . [(\nabla_u g)(\xi, \xi) + 2.g(\overline{A}_u \xi, \xi)] = f(u).l_\xi^2 \tag{36}$$

with $\overline{A}_u = \overline{g}(C(u)) = g(b,u).\overline{g}(^F\omega) + g(q,u).\overline{g}(g) - \frac{1}{2}.\overline{g}(\nabla_u g)$ is fulfilled identically for $\forall f(u) \in C^r(M)$. For two null contravariant vector fields ξ and η $[l_\xi^2 = 0,\ l_\eta^2 = 0]$ under a conformal transport the relation

$$l_\xi . l_\eta . u[\cos(\xi, \eta)] = (\nabla_u g)(\xi, \eta) + g(\overline{A}_u \xi, \eta) + g(\xi, \overline{A}_u \eta) = 0$$

is also identically fulfilled.

Remark. Contravariant null vector fields fulfil also identically the conditions for Fermi-Walker transports.

Therefore, *every contravariant null vector field ξ obeys automatically the conditions for a conformal transport.* Moreover, *the (right) angle between two contravariant null vector fields ξ and η is automatically preserved under a conformal transport.*

If the contravariant vector field u is a null vector field $[e = g(u,u) = 0]$ we cannot choose b as $b = \frac{1}{e}.u$ but we can consider it as $b = u$. In such a case $g(b,u) = g(u,u) = 0$ and $\overline{A}_u$ will have the form

$$\overline{A}_u = g(q,u).\overline{g}(g) - \frac{1}{2}.\overline{g}(g) = \overline{g}(C(u)) \ . \tag{37}$$

If we define further q as a contravariant null vector field $[g(q,q) = 0]$, then q could be written in the form $q = \alpha.u$ $[\alpha \in C^r(M)]$ and $\overline{A}_u$ will have the form

$$\overline{A}_u = -\frac{1}{2}.\overline{g}(\nabla_u g) \tag{38}$$

identical with the form of $\overline{A}_u$ for a Fermi-Walker transport of type B.

5 The length of a contravariant vector field as harmonic oscillator over $(\overline{L}_n, g)$-spaces

Let us now try to find out a solution of the following problem: under which conditions the length l_ξ of a non-null vector field ξ moving under conformal transport along a contravariant vector field u could fulfil the equation of a harmonic oscillator in the form

$$\frac{d^2 l_\xi}{d\tau^2} + \omega_0^2 . l_\xi = 0 \ , \qquad \omega_0^2 = \text{ const. } \geq 0 \ , \ u^i = \frac{dx^i}{d\tau} \ , \quad u = \frac{d}{d\tau} \ . \tag{39}$$

Lets a contravariant non-null vector field ξ be given, moving under conformal transport along a trajectory $x^k(\tau)$ with the tangential vector $u = \frac{d}{d\tau}$ in a $(\overline{L}_n, g)$-space. The change of the length of ξ along the trajectory is given as [see (9) and (22)]

$$\frac{dl_\xi}{d\tau} = g(q, u).l_\xi \ , \qquad \xi, q, u \in T(M) \ . \tag{40}$$

Since q is an arbitrary given contravariant vector field we can specify its structure in a way allowing us to consider $g(q, u)$ as an invariant scalar function depending only of the parameter τ of the trajectory $x^k(\tau)$. A possible choice of q fulfilling this precondition is

$$q = \omega(\tau).\frac{1}{e}.u \ , \qquad \omega(\tau) = \omega(x^k(\tau)) \in C^r(M) \ , \quad r \geq 1, \quad e = g(u, u) \neq 0 \ . \tag{41}$$

Then we have

$$\frac{dl_\xi}{d\tau} = \omega(\tau).l_\xi \tag{42}$$

Remark. The solution of this equation for l_ξ can be found in the form

$$l_\xi = l_{\xi 0}.\exp(\int \omega(\tau).d\tau) \ , \qquad l_{\xi 0} = \text{ const.}, \quad l_\xi > 0 \ , \tag{43}$$

where $d(\log l_\xi) = \omega(\tau).d\tau$, $\log l_\xi = \int \omega(\tau).d\tau + C_0$, $C_0 = \text{const.}$

After differentiation of the expression (42) along τ and after further use of the same expression we obtain

$$\frac{d^2 l_\xi}{d\tau^2} = \frac{d\omega(\tau)}{d\tau}.l_\xi + \omega(\tau).\frac{dl_\xi}{d\tau} = \frac{d\omega(\tau)}{d\tau}.l_\xi + [\omega(\tau)]^2.l_\xi =$$

$$= \left\{ \frac{d\omega(\tau)}{d\tau}. + [\omega(\tau)]^2 \right\}.l_\xi \ . \tag{44}$$

The equation for $d^2 l_\xi / d\tau^2$ could be written in the form

$$\frac{d^2 l_\xi}{d\tau^2} - \left\{ \frac{d\omega(\tau)}{d\tau}. + [\omega(\tau)]^2 \right\}.l_\xi = 0 \ . \tag{45}$$

It could have the form of an oscillator equation if the arbitrary given until now function $w(\tau)$ fulfils the equation

$$\frac{d\omega(\tau)}{d\tau}. + [\omega(\tau)]^2 = -\ \omega_0^2 \leq 0 \tag{46}$$

and could be determined by means of this (additional) condition. The last equation is a Riccati equation [6]. By the use of the substitution $u' = \omega.u$,

$u' = du/d\tau$, it could be written in the form of a homogeneous harmonic oscillator equation for u

$$u'' + \omega_0^2.u = 0 \ , \tag{47}$$

where

$$\omega(\tau) = \frac{u'(\tau)}{u(\tau)} \ , \quad \omega' = -\frac{(u')^2}{u^2} + \frac{u''}{u} \ , \quad \omega' + \omega^2 = -\omega_0^2 \ .$$

For $\omega_0^2 \geq 0$ (47) has the solutions:
(a) $\omega_0^2 = 0 : u' = C_2.\tau \ , \quad u = C_1 + C_2.\tau \ , C_1, C_2 =$ const.
(b) $\omega_0^2 > 0 : u = C_1.\cos\omega_0.\tau + C_2.\sin\omega_0.\tau \ , C_1, C_2 =$ const.
The solution for ω will have the forms:
(a) $\omega_0^2 = 0$:

$$\omega(\tau) = \frac{C_2.\tau}{C_1 + C_2.\tau} = \frac{\tau}{\frac{C_1}{C_2} + \tau} = \frac{\tau}{a + \tau} \ , \quad a = \frac{C_1}{C_2} = \text{const.}, \quad C_2 \neq 0 \ . \tag{48}$$

(b) $\omega_0^2 > 0 : u = C_1.\cos\omega_0.\tau + C_2.\sin\omega_0.\tau, \ u' = \omega_0.[C_2.\cos\omega_0.\tau - C_1.\sin\omega_0.\tau]$, C_1, $C_2 =$ const.,

$$\omega(\tau) = \frac{u'}{u} = \omega_0.\frac{C_2.\cos\omega_0.\tau - C_1.\sin\omega_0.\tau}{C_1.\cos\omega_0.\tau + C_2.\sin\omega_0.\tau} = \omega_0.\frac{1 - \frac{C_1}{C_2}.\tan\omega_0.\tau}{\frac{C_1}{C_2} + \tan\omega_0.\tau} \ , \quad C_2 \neq 0 \ , \tag{49}$$

$$\omega(\tau) = \omega_0.\frac{1 - a.\tan\omega_0.\tau}{a + \tan\omega_0.\tau} \ , \quad a = \frac{C_1}{C_2} = \text{ const.} \tag{50}$$

(b_1) For $C_2 = 0 : \omega(\tau) = -\omega_0.\tan\omega_0.\tau$.
(b_2) For $C_1 = 0 : \omega(\tau) = \omega_0.\cot\omega_0.\tau = \omega_0/\tan\omega_0.\tau$.

Therefore, if l_ξ should be a harmonic oscillator in a $(\overline{L}_n, g)$-space with a given constant frequency ω_0 the function $\omega(\tau)$ has to obey a Riccati equation determining its structure. In this case, the length l_ξ will be a homogeneous harmonic oscillator obeying at the same time the equation

$$\frac{dl_\xi}{d\tau} = \omega_0.\frac{1 - a.\tan\omega_0.\tau}{a + \tan\omega_0.\tau}.l_\xi \ , \qquad a = \frac{C_1}{C_2} = \text{ const.} \neq 0 \ , \tag{51}$$

which appears in general as a solution of the homogeneous harmonic oscillator's equation

$$\frac{d^2 l_\xi}{d\tau^2} + \omega_0^2.l_\xi = 0 \ , \qquad \omega_0^2 = \text{ const. } \geq 0 \ .$$

If we could simulate by the use of an appropriate experimental device the change $(dl_\xi/d\tau)$ as given in its equation (51), then we can be sure that l_ξ will swing as a harmonic oscillator with the frequency ω_0 over a $(\overline{L}_n.g)$-space. This

could allow us to investigate experimentally the influence of physical interactions on the length l_ξ of a vector field ξ moving under a conformal transport over a $(\overline{L}_n, g)$-space. On the other hand, if we can register (as an observer) from our basic trajectory a change of l_ξ in accordance with the equation for $(dl_\xi/d\tau)$, then we can conclude that l_ξ moves as harmonic oscillator under the external (or internal) forces. Since the considered here problem is related to the kinetic aspect of the motion of l_ξ, the dynamic aspect should be introduced by means of a concrete field (gravitational) theory, which is not a subject of the above considerations.

5.1 Geometrical and physical interpretation of the function $\omega(\tau)$

The covariant derivative $\nabla_u\xi$ of a contravariant vector field ξ along a contravariant vector field u in a $(\overline{L}_n, g)$-space can be represented in the form [7]

$$\nabla_u\xi = \frac{\overline{l}}{e}.u + \overline{g}[h_u(\nabla_u\xi)] = \frac{\overline{l}}{e}.u + {}_{rel}v \ , \qquad \overline{l} = g(\nabla_u\xi, u) \tag{52}$$

where

$${}_{rel}v = \overline{g}[h_u(\nabla_u\xi)] = \overline{g}(h_u)(\frac{l}{e}.a - \pounds_\xi u) + \overline{g}[d(\xi)] \ , \quad h_u = g - \frac{1}{e}.g(u) \otimes g(u) \ , \tag{53}$$

$$a = \nabla_u u \ , \qquad d = \sigma + \omega + \frac{1}{n-1}.\theta.h_u \ , \ \ g(\nabla_u\xi, u) = ul - (\nabla_u g)(\xi, u) - g(\xi, a) \ . \tag{54}$$

σ is the shear velocity tensor (shear), ω is the rotation velocity tensor (rotation), θ is the expansion velocity invariant (expansion), d is the deformation velocity tensor (deformation), $l = g(\xi, u)$, ${}_{rel}v$ is the relative velocity. If ξ is an orthogonal to u vector field $[\xi = \xi_\perp = \overline{g}[h_u(\xi)]]$, then $l = 0$ and under the additional precondition $\pounds_\xi u = -\pounds_u\xi = 0$ the expression for ${}_{rel}v$ will take the form

$${}_{rel}v = \overline{g}[d(\xi_\perp)] = \overline{g}[\sigma(\xi_\perp)] + \overline{g}[\omega(\xi_\perp)] + \frac{1}{n-1}.\theta.\xi_\perp \ . \tag{55}$$

The rate of change of the length $l_{\xi_\perp}$ of the vector field $\xi_\perp$ (along the vector field $u = \frac{d}{d\tau}$) in $\overline{U}_n$- or $\overline{V}_n$-spaces $[\nabla_u g = 0$ for $\forall u \in T(M)]$, under the conditions $l = 0$ $[\xi = \xi_\perp = \overline{g}[h_u(\xi)]]$ and $\pounds_u\xi = 0$, can be found in the form

$$ul_{\xi_\perp} = \frac{dl_{\xi_\perp}}{d\tau} = \pm\frac{1}{l_{\xi_\perp}}.d(\xi_\perp, \xi_\perp) = \pm\frac{1}{l_{\xi_\perp}}.\sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1}.\theta.l_{\xi_\perp} \ , \qquad l_{\xi_\perp} \neq 0 \ . \tag{56}$$

Remark. The sign $\pm$ depends on the sign of the metric g (for $n = 4$, sign $g = \pm 2$).

If a U_n- or a V_n-space admits a shear-free non-null auto-parallel vector field u ($\nabla_u u = a = 0$), then $\sigma = 0$ and

$$\nabla_u \xi_\perp = {}_{rel}v = \overline{g}[\omega(\xi_\perp)] + \frac{1}{n-1}.\theta.\xi_\perp = \overline{g}[\omega(\xi_\perp)] + \frac{1}{n-1}.\theta.\overline{g}[g(\xi_\perp)] = \quad (57)$$

$$= \overline{g}[C(u)(\xi_\perp)] = \overline{A}_u \xi_\perp \ , \qquad C(u) = \omega + \frac{1}{n-1}.\theta.g \ . \quad (58)$$

$$\frac{dl_{\xi_\perp}}{d\tau} = \frac{1}{n-1}.\theta.l_{\xi_\perp} \ . \quad (59)$$

Therefore, the vector field $\xi_\perp$ undergoes a conformal transport along u. A comparison with (42) show us that in this case we can choose the arbitrary given function $\omega(\tau)$ as

$$\omega(\tau) = \frac{1}{n-1}.\theta \ . \quad (60)$$

The last fact leads to the conclusion that $\omega(\tau)$ could be related to the expansion velocity θ in a $\overline{U}_n$- or a $\overline{V}_n$-space. In $(\overline{L}_n, g)$-spaces it could preserve its interpretation.

6 Conclusions

In the present paper we have considered types of transports more general than the Fermi-Walker transports. They are called conformal transports over $(\overline{L}_n, g)$-spaces. In an analogous way as in the case of Fermi-Walker transports a conformal covariant differential operator and its corresponding conformal derivative are determined and discussed over $(\overline{L}_n, g)$-spaces. Different special types of conformal transports are considered inducing also Fermi-Walker transports for orthogonal vector fields as special cases. Conditions under which the length of a non-null contravariant vector field will swing as a homogeneous harmonic oscillator with given frequency are established. The results obtained regardless of any concrete field (gravitational) theory could have direct applications in such types of theories.

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